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# Complex multiplication, rationality and mirror symmetry for Abelian varieties

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**Abstract**

We show that complex multiplication on Abelian varieties is equivalent to the existence of a constant rational Kähler metric. We give a sufficient condition for a mirror of an Abelian variety of CM-type to be of CM-type. We also study the relationship between complex multiplication and rationality of a toroidal lattice vertex algebra.


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## 1. Introduction and results

This article is inspired by Gukov–Vafa's paper [6], where they shared their insight into the interplay between rational conformal field theories (CFTs), mirror symmetry and complex multiplication on Calabi–Yau varieties. Here we study the case of complex Abelian varieties of arbitrary dimension, where the above notions find a sound mathematical foundation.

Complex multiplication (CM) on Abelian varieties has been extensively studied in geometry as well as in number theory (see e.g. [23,14]). Roughly speaking, Abelian varieties of CM-type have by definition the biggest endomorphism algebra (see Definition 2.1). It is a property which solely depends on the complex structure of the Abelian variety. For example, in dimension one, an elliptic curve $E \cong \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ is of CM-type if and only if $\tau$ lies in an imaginary quadratic number field $\mathbb{Q}(\sqrt{-D})$. In physics, toroidal CFTs are also very familiar objects (see e.g. [25, 3] and the references therein). From a physicist's point of view, a CFT is called rational, if its partition function can be written as a finite sum of the product of holomorphic and anti-holomorphic characters (see [6, Section 2]), and this is the case when its chiral part is maximal.

The interplay between CM and rational CFT having a real 2-torus as target space is already known in Moore's paper [16, Section 10], and it is generalized by Wendland to real tori of arbitrary dimension. We can rephrase her

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Theorem 4.5.5 in [25] as follows: (a) A real torus $\mathbb{T}$ admits a rational constant metric $G$ if and only if $\mathbb{T}$ can be endowed with a complex structure $I$ such that the complex torus $(\mathbb{T}, I)$ is isogenous to a product of elliptic curves of CM-type. (b) A CFT $\mathcal{C}(\mathbb{T}, G, B)$ associated to a real torus $\mathbb{T}$ endowed with a constant metric $G$ and a B-field $B \in H^2(\mathbb{T}, \mathbb{R})$ is rational if and only if both $G$ and $B$ are rational. (c) Combining (a) and (b) one can say that a real torus $\mathbb{T}$ admits a rational CFT $\mathcal{C}(\mathbb{T}, G, B)$ if and only if $\mathbb{T}$ can be endowed with a complex structure $I$ such that $(\mathbb{T}, I)$ is isogenous to a product of elliptic curves of CM-type.

In [6] Gukov and Vafa relate this to mirror symmetry. They make the following observation: Let $(E' \cong \mathbb{C}/\mathbb{Z} \oplus \rho\mathbb{Z}, G', B')$ be a mirror elliptic curve of $(E \cong \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}, G, B)$, where $G$ is a constant Kähler metric. Then the $N = 2$ CFT $\mathcal{C}(E, G, B)$ is rational if and only if both $E'$ and $E$ are of CM-type over the same imaginary quadratic field, i.e. $\tau, \rho \in \mathbb{Q}(\sqrt{-D})$, in this case, $E'$ and $E$ are in particular isogenous. Note that the difference between this statement and Wendland's result is that Gukov and Vafa start with an elliptic curve with a *given* complex structure endowed with a *Kähler* metric, and use an additional $N = 2$ structure on the CFT to encode the complex structure of the target space, and hence link the complex geometry (e.g. CM) of the mirror pair.

One then naturally asks whether similar relations hold for Abelian varieties of arbitrary dimension, where, along with CM and mirror symmetry, also CFTs and nonlinear sigma model received a mathematical treatment, this is given by Kapustin and Orlov in terms of vertex algebras (see [13]). Note that in their work, it is already apparent that mirror symmetry of the target space, i.e. the complex tori, is equivalent to mirror symmetry of the $N = 2$ vertex algebras (see Theorem 4.5 in [13]). We shall use a slightly different construction, which we call a *lattice vertex algebra* (for in a special case, it reduces to the the well-known lattice vertex algebra constructed in [11, Section 5.4]). It turns out that it is isomorphic to the Kapustin–Orlov vertex algebra (see Appendix), but more suitable for our discussion of CM and rationality. Given these properly defined notions, we formulate the question which we fully answer in this article:

> *Let $(X', G', B')$ be a mirror partner of $(X, G, B)$, where $X$ is an Abelian variety endowed with a constant Kähler metric $G$ and a B-field in $H^2(X, \mathbb{R})$. Is the $N = 2$ lattice vertex algebra $V(X, G, B)$ rational if and only if $X$ and $X'$ are isogenous and both of CM-type?* (Q)

Let us now explain our main results. The following theorem shows that CM, which is *a priori* determined solely by the complex structure, turns out to be equivalent to the rationality of a Kähler metric.

**Theorem 2.5.** *An Abelian variety $X$ is of CM-type if and only if $X$ admits a constant rational Kähler metric.*

(A rational Kähler metric is a Kähler metric which takes only rational values on the lattice $\Gamma$ of $X$.) This theorem is apparently independent of mirror symmetry and CFT, but it plays an important rôle in the relation between CM and the rationality of $N = 2$ lattice vertex algebras. This will be evident through the following theorem.

**Theorem 5.7.** *The $N = 2$ lattice vertex algebra $V(T, G, B)$ associated to a complex torus $T$ endowed with a constant Kähler metric $G$ and a B-field is rational if and only if $G$ and $B$ are both rational.*

A few comments are due here. The rationality of a $N = 2$ lattice vertex algebra is defined on the underlying lattice vertex algebra (i.e. without the $N = 2$ structure, see Definition 5.5). Hence forgetting the $N = 2$ structure and the complex structure of $T$, this theorem is in complete accordance with Wendland's result mentioned earlier. Moreover, our definition is also in accordance with the rationality defined in terms of the partition function, this is exhibited in Remarks 5.2 and 5.8. Combining the last two theorems we have

**Corollary 5.12.** *An Abelian variety $X$ is of CM-type if and only if $X$ admits a rational $N = 2$ lattice vertex algebra $V(X, G, B)$.*

Now we make the link between CM and mirror symmetry for tori. In Section 3 we define mirror symmetry in terms of generalized Kähler structures (GKS) (see Definition 3.2) which have the advantage of treating the complex and the symplectic structures of a triple $(T, G, B)$ on equal footing. For $(T, G, B)$ one can define the following GKS $(\mathcal{I}, \mathcal{J})$:

$$\mathcal{I} := \begin{pmatrix} I & 0 \\ BI + I^t B & -I^t \end{pmatrix} \quad \text{and} \quad \mathcal{J} := \begin{pmatrix} \omega^{-1} B & -\omega^{-1} \\ \omega + B\omega^{-1} B & -B\omega^{-1} \end{pmatrix},$$

where $\omega$ is the Kähler form $G(\cdot, I\cdot)$ (see Definition 3.5 for more details). Note that the composition $\mathcal{I}\mathcal{J}$ of such an induced GKS is defined over $\mathbb{Q}$ if and only if both $G$ and $B$ are rational (see Definition 3.7 and Lemma 3.8(ii)). A pair

$(T, \mathcal{I}, \mathcal{J})$ and $(T', \mathcal{I}', \mathcal{J}')$ is called a mirror pair if there is a mirror map which exchanges $\mathcal{I}$ with $\mathcal{J}'$ and $\mathcal{J}$ with $\mathcal{I}'$ (see Definition 3.4). We find a sufficient condition for CM to be transmitted to mirror partners:

**Theorem 3.11.** *Let $(X, G, B)$ and $(X', G', B')$ be mirror Abelian varieties. Suppose that $X$ is of CM-type. If both $G$ and $B$ are rational, then $X$ and $X'$ are isogenous. In particular, $X'$ is also of CM-type.*

The converse of this theorem however does not hold:

**Proposition 4.1.** *There are mirror Abelian varieties $(X, G, B)$ and $(X', G', B')$, such that $X$ and $X'$ are isogenous and of CM-type, but neither $\mathcal{I}\mathcal{J}$ nor $\mathcal{I}'\mathcal{J}'$ is defined over $\mathbb{Q}$, where $(\mathcal{I}, \mathcal{J})$ and $(\mathcal{I}', \mathcal{J}')$ denote their induced GKS.*

Combining all the results from the above, we now formulate our answer to the question (Q):

**Corollary 5.13.** *Let $(X, G, B)$ and $(X', G', B')$ be mirror Abelian varieties. If the $N = 2$ lattice vertex algebra $V(X, G, B)$ is rational, then $X$ and $X'$ are isogenous and both of CM-type. Conversely, however, there exist mirror Abelian varieties $(X, G, B)$ and $(X', G', B')$ such that $X$ and $X'$ are isogenous and both of CM-type, but neither $V(X, G, B)$ nor $V(X', G', B')$ is rational.*

This paper is organized as follows: Section 2 presents the proof of Theorem 2.5 which is a pure geometric result. Section 3 formulates mirror symmetry for tori and we prove Theorem 3.11. In Section 4 we give an explicit counter-example to the converse of Theorem 3.11, proving Proposition 4.1. Section 5 deals with vertex algebras. We construct the lattice vertex algebra, which is isomorphic to the Kapustin–Orlov vertex algebra (shown in Appendix), and we define the notion of rationality, and we prove Theorem 5.7, Corollaries 5.12 and 5.13.

## 2. Complex multiplication and rational Kähler metric

The aim of this section is to prove Theorem 2.5. Let us first explain when a constant Kähler metric is called rational. If we identify the tangent space of a complex torus $T = \mathbb{C}^g/\Gamma$ with $\Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$, then the complex structure $I$ of $T$ can be considered as an endomorphism of $\Gamma_{\mathbb{R}}$ with $I^2 = -\mathrm{Id}$. A constant Kähler metric $G$ is a positive definite bilinear form on $\Gamma_{\mathbb{R}} \times \Gamma_{\mathbb{R}}$, which is compatible with $I$, i.e. $G(I\cdot, I\cdot) = G(\cdot, \cdot)$. If $G$ takes only rational values on $\Gamma \times \Gamma$, then we call it *rational*. For complex multiplication we adopt Mumford's definition [18, Section 2]:

**Definition 2.1.** A simple Abelian variety $X$ of dimension $g$ is of CM-type over $K$ if there is an embedding $K \hookrightarrow \mathrm{End}_{\mathbb{Q}}(X) := \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ of an algebraic number field $K$ of degree $2g$ over $\mathbb{Q}$ into the endomorphism algebra of $X$. More generally, an (not necessarily simple) Abelian variety $X$ is of CM-type if $X$ is isogenous to a product of simple Abelian varieties of CM-type.

One can show (see [23, Section 5], [18, Section 2])

**Proposition 2.2.** (i) *If a* simple *Abelian variety $X$ of dimension $g$ is of CM-type over $K$, then $K$ is necessarily a CM-field (i.e. a totally complex quadratic extension of a totally real number field) of rank $2g$ over $\mathbb{Q}$.*
(ii) *More generally, an (not necessarily simple) Abelian variety $X$ of dimension $g$ is of CM-type if and only if $\mathrm{End}_{\mathbb{Q}}(X)$ contains a commutative semi-simple algebra of rank $2g$ over $\mathbb{Q}$.*
(iii) *If $\mathrm{End}_{\mathbb{Q}}(X)$ of an Abelian variety $X$ of dimension $g$ contains a number field of degree $2g$ over $\mathbb{Q}$, then $X$ is isogenous to a product $B \times \cdots \times B$ with a simple Abelian variety $B$ of CM-type.*

**Remark 2.3.** Definition 2.1 is stronger than the notion of admitting complex multiplication in [6]. There is a complex torus $T = \mathbb{C}^g/(\mathbf{1}\ \mathcal{T})\mathbb{Z}$ which is said to be "admit complex multiplication" if there is a nontrivial endomorphism $A \in GL(g, \mathbb{C})$, such that there exist integer matrices $M', N', M, N$ and $N$ has rank $g$ and $A = M + N\mathcal{T}$ and $\mathcal{T}A = M' + N'\mathcal{T}$. For example, if $\mathrm{End}_{\mathbb{Q}}(X)$ contains a number field $\mathbb{Q}(\xi)$ of rank $2g = 2\dim X$, then an integral multiple of the multiplication by $\xi$ on $\mathbb{C}^g$ would satisfy Gukov–Vafa's condition. However, one can show that the converse does not hold.

Next we recall a few general facts which we will need in what follows, e.g. the construction of simple Abelian varieties of CM-type and the characterization of complex multiplication by the Hodge group.

It can be shown that any simple Abelian variety of CM-type over a CM-field $K$ is isogenous to an Abelian variety constructed in the following way (see [23, Section 6] or [19, Chap. IV]). Let $K$ be a quadratic extension of a real field $K_0$ and denote by $\mathcal{O}_K$ its ring of integers. Let $\Phi = \{\sigma_1, \dots, \sigma_g\}$ be a CM-type, i.e. $\Phi \cup \bar{\Phi}$ is the whole set of the embeddings of $K$ into $\mathbb{C}$. One obtains a complex torus $\mathbb{C}^g/\Phi(\mathcal{O}_K)$, where $\Phi(\mathcal{O}_K)$ is the image of the embedding of $\mathcal{O}_K$ into $\mathbb{C}^g$ given by $x \mapsto (\sigma_1 x, \dots, \sigma_g x)$. There is always a Riemann form, which makes this complex torus into an Abelian variety. Indeed, there exists an element $\beta \in \mathcal{O}_K$ such that $K = K_0(\beta)$ and $-\beta^2$ is totally positive and the imaginary part satisfies $\operatorname{Im}\sigma_j(\beta) > 0, \forall \sigma_j \in \Phi$. The Riemann form on the tangent space is defined as

$$E(z, w) = \sum_{j=1}^{g} \sigma_j(\beta) \left( \sigma_j(z)\overline{\sigma_j(w)} - \overline{\sigma_j(z)}\sigma_j(w) \right). \tag{1}$$

On $\mathcal{O}_K$, it takes values in $\mathbb{Q}$:

$$E(a_1, a_2) = \operatorname{Tr}_{K/\mathbb{Q}}(\beta a_1 \bar{a}_2) \in \mathbb{Q}, \quad \forall a_1, a_2 \in \mathcal{O}_K.$$

The thus-constructed Abelian variety is simple if and only if there is *no* proper subfield $L$ of $K$ with the properties (a) $L$ is a purely complex quadratic extension of $L \cap K_0$ and (b) if $\sigma_i|_{L\cap K_0} = \sigma_j|_{L\cap K_0}$ then $\sigma_i|_L = \sigma_j|_L, \forall \sigma_i, \sigma_j \in \Phi$.

As to the Hodge group, it is one of the main tools to study Hodge structures. In the case of Abelian varieties, the Hodge structure is of weight one and the Hodge group is defined to be the smallest algebraic subgroup of $GL(\Gamma_{\mathbb{Q}})$ defined over $\mathbb{Q}$, whose $\mathbb{R}$-points include the image of the unit circle $S^1$ under the map

$$h : S^1 \longrightarrow SL(\Gamma_{\mathbb{R}}), \qquad x + yi \longmapsto x + yI$$

i.e. $h(S^1) \subset \operatorname{Hg}(X)(\mathbb{R})$. Moreover, Hodge groups are reductive and connected (see [17]). There is the following characterization of the complex multiplication. The equivalence between (ii) and (iii) below is also well known to the experts. Nevertheless, since we could not find an explicit proof in the literature, we include a complete proof for the readers' convenience.

**Proposition 2.4.** *Let $X$ be an Abelian variety. Then the following conditions are equivalent:*

(i) *$X$ is of CM-type.*
(ii) $\operatorname{Hg}(X)$ *is commutative.*
(iii) $\operatorname{Hg}(X)(\mathbb{R})$ *is compact.*

**Proof.** (i) $\Leftrightarrow$ (ii): See [18, Section 2] or [15, Prop. 17.3.5].

(ii) $\Leftrightarrow$ (iii): On $\operatorname{Hg}(X)(\mathbb{R})$ we have the conjugation by $h(i)$

$$\begin{aligned} &\operatorname{Ad} h(i) : \operatorname{Hg}(X)(\mathbb{R}) \longrightarrow \operatorname{Hg}(X)(\mathbb{R}) \\ &M \longmapsto h(i)Mh(i)^{-1}, \end{aligned}$$

which is a Cartan involution (see [2, Section 2]). If $\operatorname{Hg}(X)$ is commutative, then $\operatorname{Ad} h(i)$ is just the identity map, and hence $\operatorname{Hg}(X)(\mathbb{R})$ is compact.

Conversely, if $\operatorname{Hg}(X)(\mathbb{R})$ is compact, then the identity map is a Cartan involution. By [21, Chap.1 Cor. 4.3], any two Cartan involutions of a connected reductive real algebraic group $\mathcal{G}$ are conjugate to each other by an inner automorphism of $\mathcal{G}$. Hence we have $\operatorname{Ad} h(i) = \operatorname{Id}$ on $\operatorname{Hg}(X)(\mathbb{R})$. There are at least two different arguments to finish the proof from the above.

(a) This means that $h(i)$ lies in the centralizer $C(\operatorname{Hg}(X)(\mathbb{R}))$ of $\operatorname{Hg}(X)(\mathbb{R})$ in $SL(\Gamma_{\mathbb{R}})$. Suppose we have already shown that one in fact has

$$h(i) \in C(\operatorname{Hg}(X))(\mathbb{R}), \tag{2}$$

then the whole image $h(S^1)$ lies in $C(\operatorname{Hg}(X))(\mathbb{R})$. Indeed, since the action $h$ on $\Gamma_{\mathbb{R}}$ is linear, we have for any $M \in \operatorname{Hg}(X)$:

$$h(x + yi)M = h(x)M + h(yi)M = Mh(x) + Mh(yi) = Mh(x + yi).$$

Since $C(\operatorname{Hg}(X))$ is defined over $\mathbb{Q}$, for $\operatorname{Hg}(X)$ is so, this shows that $C(\operatorname{Hg}(X))$ contains $\operatorname{Hg}(X)$. Hence $\operatorname{Hg}(X)$ is commutative.

It remains to prove (2). Denote the stabilizer of $h(i)$ in $\operatorname{Hg}(X)(\mathbb{C})$ by

$$H := \{M \in \mathrm{Hg}(X)(\mathbb{C}) \mid h(i)M = Mh(i)\}$$

Since $H$ is defined by algebraic equations, it is a closed subgroup of $\mathrm{Hg}(X)(\mathbb{C})$. Moreover, $H$ contains $\mathrm{Hg}(X)(\mathbb{R})$. So, from the fact that the $\mathbb{R}$-points $\mathcal{G}(\mathbb{R})$ of any linear algebraic group $\mathcal{G}$ are Zariski-dense in $\mathcal{G}(\mathbb{C})$, it follows that

$$H \supset \overline{\mathrm{Hg}(X)(\mathbb{R})} = \mathrm{Hg}(X)(\mathbb{C}).$$

Hence $H = \mathrm{Hg}(X)(\mathbb{C})$, and $h(i) \in C(\mathrm{Hg}(X)(\mathbb{C}))$. Since $\mathbb{C}$ is algebraically closed, we conclude that $C(\mathrm{Hg}(X)(\mathbb{C})) = C(\mathrm{Hg}(X))(\mathbb{C})$, which implies (2).

(b) From $\mathrm{Ad}\,h(i) = \mathrm{Id}$ on $\mathrm{Hg}(X)(\mathbb{R})$ it follows that for any $\mathbb{Q}$-point $N \in \mathrm{Hg}(X)(\mathbb{Q})$ we have $NI = IN$, and hence $\mathrm{Hg}(X)(\mathbb{Q}) \subset \mathrm{End}_{\mathbb{Q}}(X)$. On the other hand, we have

$$\mathrm{End}_{\mathbb{Q}}(X) = \mathrm{End}(\Gamma_{\mathbb{Q}})^{\mathrm{Hg}(X)}$$

(see [15, Prop. 17.3.4]), thus $\mathrm{Hg}(X)(\mathbb{Q})$ is fixed under the conjugation with itself, whence $\mathrm{Hg}(X)(\mathbb{Q})$ is commutative.

In order to conclude that $\mathrm{Hg}(X)$ is commutative, we have to show that $\mathrm{Hg}(X)(\mathbb{C})$ is commutative. By the above, $\mathrm{Hg}(X)(\mathbb{Q})$ lies in the center of $\mathrm{Hg}(X)(\mathbb{C})$. Recall that the center of a linear algebraic group is a closed subgroup (see in [8, Cor. Section 8.2]). By [24, Cor. 13.3.9] or [8, Thm. in Section 34.4], $\mathrm{Hg}(X)(\mathbb{Q})$ is Zariski-dense in $\mathrm{Hg}(X)(\mathbb{C})$, and therefore

$$\mathrm{Center}(\mathrm{Hg}(X)(\mathbb{C})) \supset \overline{\mathrm{Hg}(X)(\mathbb{Q})} = \mathrm{Hg}(X)(\mathbb{C}),$$

which implies the commutativity of $\mathrm{Hg}(X)(\mathbb{C})$. $\square$

Now we prove

**Theorem 2.5.** *An Abelian variety $X$ is of CM-type if and only if $X$ admits a constant rational Kähler metric.*

**Proof.** $\Leftarrow$: First suppose that $G$ is an arbitrary constant Kähler metric on $X$. Then for all $z = x + yi \in S^1$ we have

$$G(h(z)v, h(z)w) = G((x + yI)v, (x + yI)w) = (x^2 + y^2)G(v, w) = G(v, w),$$

in other words, $h(S^1) \subset O(G, \mathbb{R})$. If $G$ is moreover rational, then $O(G)$ is an algebraic group defined over $\mathbb{Q}$, whose $\mathbb{R}$-points contain $h(S^1)$. Hence $\mathrm{Hg}(X)$ is an algebraic subgroup of $O(G)$, and in particular $\mathrm{Hg}(X)(\mathbb{R}) \subset O(G, \mathbb{R})$. Therefore $\mathrm{Hg}(X)(\mathbb{R})$ is compact, and $X$ is of CM-type by Proposition 2.4.

$\Rightarrow$: If $X$ is of CM-type, then $X$ is isogenous to a product of simple Abelian varieties of CM-type by Definition 2.1. So we may assume that $X$ is simple. As mentioned before, $X$ is then isogenous to a simple Abelian variety of CM-type with a Riemann form $E$ as in (1). It allows us to define the following bilinear form on the tangent space:

$$G(z, w) := E(z, \beta w) = \mathrm{Tr}_{K/\mathbb{Q}}(-\beta^2 z\bar{w}).$$

We see that $G$ is compatible with $I$ (as $E$ is), rational, symmetric, and positive definite (as $-\beta^2$ is totally positive). Since the existence of a rational Kähler metric is preserved under isogeny, this completes the proof. $\square$

We shall give an alternative proof of the "$\Leftarrow$" direction of Theorem 2.5 for a *simple* Abelian variety $X$. It has the advantage of exhibiting more clearly how a rational metric endows $\mathrm{End}_{\mathbb{Q}}(X)$ with additional structures which force $\mathrm{End}_{\mathbb{Q}}(X)$ to be very "big". Let us first make a reminder of some general facts about $\mathrm{End}_{\mathbb{Q}}(X)$.

Due to the presence of the Rosati involution, the endomorphism algebra of a simple Abelian variety must be a division algebra of finite rank over $\mathbb{Q}$ endowed with a positive anti-involution. Recall that an involution $f \mapsto f^\sigma$ on a division algebra $A$ with center $K$ is called positive, if the quadratic form

$$\mathrm{tr}_{A|\mathbb{Q}}\, f^\sigma f := \mathrm{Tr}_{K|\mathbb{Q}}(\mathrm{tr}_{A|K}\, f^\sigma f) \tag{3}$$

is positive definite, where $\mathrm{tr}_{A|K}$ denotes the reduced trace of $A$ over $K$, and $\mathrm{Tr}_{K|\mathbb{Q}}$ denotes the usual trace for the field extension $K|\mathbb{Q}$. Albert gave the classification of such division algebras $A$ (see [15, Thm 5.5.3, Lemma 5.5.4 and Prop. 5.5.5]):

I. $A$ = totally real number field, left invariant by the positive anti-involution.
II. $A$ = totally indefinite quaternion algebra, there is an element $a \in A$ whose square $a^2$ is in its center $K$ and is totally negative (i.e. is a negative real number under any embedding $K \hookrightarrow \mathbb{R}$), such that the positive anti-involution $f \mapsto f^\sigma$ is given by $f^\sigma = a(\mathrm{tr}_{A|K}\, f - f)a^{-1}$.

III. $A =$ totally definite quaternion algebra, and $f \mapsto f^{\sigma}$ is given by $f^{\sigma} = \mathrm{tr}_{A|K} f - f$.

The first three algebras are of *the first kind*, i.e. the center is a totally real number field and coincides with the subfield fixed by the involution.

IV. The center $K$ of $A$ is a CM-field. The positive anti-involution restricted to $K$ is the complex conjugation.

Let us put $F := \mathrm{End}_{\mathbb{Q}}(X)$ and denote by $\omega_0$ a (rational) polarization of $X$, which always exists, since $X$ is algebraic. The index 0 is to distinguish it from the Kähler form $\omega = GI$, which in general is not rational. Further we denote by $f \mapsto f'$ the Rosati involution with respect to $\omega_0$ and by $G_0$ the Kähler metric associated to $\omega_0$.

The presence of a rational Kähler metric induces two new structures on $F$:

- A linear map $\eta \in \mathrm{End}(\Gamma_{\mathbb{Q}})$ determined by

$$G(\cdot, \cdot) = \omega_0(\eta \cdot, \cdot). \tag{4}$$

- An involution $f \mapsto f^G$ on $\mathrm{End}(\Gamma_{\mathbb{Q}})$ defined by

$$G(fv, w) = G(v, f^G w).$$

They have the following properties:

**Lemma 2.6.** (i) $\eta \in F$.
(ii) $\eta' = -\eta$.
(iii) $f \mapsto f^G$ *defines a positive anti-involution on* $F$.
(iv) *The involution* $f \mapsto f^G$ *and the Rosati involution are conjugate to each other by* $\eta$, *i.e.* $f^G = \eta^{-1} f' \eta$, $\forall f \in F$.

**Proof.** (i) Since $\omega_0$ and $G$ are compatible with $I$, we have for all $v, w \in \Gamma_{\mathbb{Q}}$:

$$\omega_0(\eta I v, w) = G(Iv, w) = -G(v, Iw) = -\omega_0(\eta v, Iw) = \omega_0(I\eta v, w),$$

hence $\eta I = I\eta$, i.e. $\eta \in F$.

(ii) Since $\omega_0(\eta v, w) = \omega_0(v, \eta' w)$, it suffices to show $\omega_0(\eta v, w) = -\omega_0(v, \eta w)$ for all $v, w \in \Gamma_{\mathbb{Q}}$. This follows from

$$\begin{aligned} \omega_0(\eta^{-1} v, w) &= -\omega_0(w, \eta^{-1} v) \\ &= -G(\eta^{-1} w, \eta^{-1} v) \\ &= -G(\eta^{-1} v, \eta^{-1} w) \\ &= -\omega_0(v, \eta^{-1} w). \end{aligned}$$

(iii) If $f \in F$, i.e. $fI = If$, then $(fI)^G = (If)^G$ and hence $I^G f^G = f^G I^G$. As $I^G = -I$ we get $If^G = f^G I$, i.e. $f^G \in F$. Next we show that $f \mapsto f^G$ defines a positive anti-involution, i.e. $\mathrm{tr}_{F|\mathbb{Q}} f^G f > 0$ for all $f \neq 0 \in F$. Since $F$ acts on $\Gamma_{\mathbb{Q}}$, one has $\Gamma_{\mathbb{Q}} \cong F^m$, and $f$ acts on $F^m$ by left multiplication on each component. On the other hand, the action of $F$ on itself by the left multiplication has trace $d \cdot \mathrm{tr}_{F|K} f$ over its center $K$, where $\mathrm{tr}_{F|K}$ is the reduced trace and $d^2$ is the degree of $F$ over $K$. Denote by $\mathrm{Tr}\, f$ the trace of $f \in F$, when considered as an endomorphism of $\Gamma_{\mathbb{Q}}$. Then we have

$$\mathrm{Tr}\, f = m \cdot d \cdot \mathrm{Tr}_{K|\mathbb{Q}}(\mathrm{tr}_{F|K} f) = m \cdot \mathrm{tr}_{F|\mathbb{Q}} f, \quad \forall f \in F. \tag{5}$$

In an orthonormal basis with respect to $G$, $f^G$ is just the transposed matrix of $f$, hence $\mathrm{Tr}\, f^G f > 0, \forall f \neq 0 \in F$. Then (5) implies that the involution induced by $G$ is positive.

(iv) This follows immediately from

$$\omega_0(f'\eta v, w) = \omega_0(\eta v, fw) = G(v, fw) = G(f^G v, w) = \omega_0(\eta f^G v, w)$$

for all $v, w \in \Gamma_{\mathbb{Q}}$, which yields $\eta f^G = f'\eta$. $\square$

We shall use Lemma 2.6 to eliminate the algebras of the first kind, and then show that the CM-field $K$ of Type IV is necessarily of maximal rank, i.e. $2g$ over $\mathbb{Q}$, which implies that $X$ is of CM-type.

*Second proof of* "$\Leftarrow$" *of* Theorem 2.5 *for a simple Abelian variety* $X$. Type I is already made impossible by (i) and (ii) of Lemma 2.6. On Type III algebras there is a unique positive anti-involution (see [22, Prop. 3]), hence $f' = f^G$. Then Lemma 2.6(iv) implies that $\eta$ lies in $K$, in contradiction with Lemma 2.6(ii).

On Type II algebras, positive anti-involutions are not unique and they are all of the form given in Albert's classification mentioned above. Let us write $f^\rho := \mathrm{tr}_{F|K} f - f$, then

$$f' = a_1 f^\rho a_1^{-1} \quad \text{and} \quad f^G = a_2 f^\rho a_2^{-1},$$

for some $a_1$ and $a_2$ in $F$. Then $f^G = a_2 a_1^{-1} f' a_1 a_2^{-1}$, and, hence, $\eta = \epsilon a_1 a_2^{-1}$ for some $\epsilon$ in $K$. On the one hand, $\eta' = -\eta$ by Lemma 2.6(ii), and on the other hand, in view of $a_i^\rho = -a_i$ (since $a_i^2 \in K$ and $a_i \notin K$), we have

$$\begin{aligned} \eta' &= \epsilon a_2^{-1'} a_1' \\ &= \epsilon a_1 (a_2^{-1})^\rho a_1^{-1} a_1 a_1^\rho a_1^{-1} \\ &= \epsilon a_1 (-a_2^{-1})(-a_1) a_1^{-1} \\ &= \epsilon a_1 a_2^{-1} \\ &= \eta. \end{aligned}$$

A contradiction.

It remains to deal with Type IV algebras. Recall that the center $K$ of $F$ is a CM-field. Let us denote by $2m$ its rank over $\mathbb{Q}$ and put $n := \frac{g}{m}$. We shall show $n = 1$, which implies that $X$ is of CM-type. Under $I$ there is the splitting

$$\Gamma_{\mathbb{C}} \xrightarrow{\sim} \Gamma_{\mathbb{C}}^{1,0} \oplus \Gamma_{\mathbb{C}}^{0,1}. \tag{6}$$

We extend the action of $F$ on $\Gamma_{\mathbb{Q}}$ $\mathbb{R}$-linearly on $\Gamma_{\mathbb{R}}$ and denote by $\rho$ its action on $\Gamma_{\mathbb{C}}^{1,0}$ under the isomorphism $\Gamma_{\mathbb{C}}^{1,0} \cong \Gamma_{\mathbb{R}}$. Since $K$ is commutative and there is an isomorphism $\Gamma_{\mathbb{Q}} \cong K^n$, the action of $K$ on $\Gamma_{\mathbb{Q}}$ diagonalizes on $\Gamma_{\mathbb{C}}$, and the diagonal entries are exactly $n$ copies of the complete set of $2m$ embeddings of $K$ into $\mathbb{C}$. The splitting (6) then implies that $\rho(K)$ even diagonalizes on $\Gamma_{\mathbb{C}}^{1,0}$, i.e. there is a complex basis $\{e_1, \ldots, e_g\}$ of $\Gamma_{\mathbb{R}}$, with respect to which, for all $x \in K$ we have $\rho(x)e_l = \rho_l(x)e_l$, where $\Psi := \{\rho_1, \ldots, \rho_g\}$ are embeddings of $K$ into $\mathbb{C}$.

We show that $\rho_l$ and $\bar{\rho}_l$ can not both belong to $\Psi$. Recall that $\eta \in K$ is the element with $G(\cdot,\cdot) = \omega_0(\eta\cdot,\cdot)$ and $\eta' = -\eta$. Thus $\rho(\eta)$ has only purely imaginary diagonal entries, as the Rosati involution restricted to $K$ is just the complex conjugation. Suppose that $\rho_2 = \bar{\rho}_1$. Since $G$ is positive definite on $\Gamma_{\mathbb{R}}$, we have

$$\begin{aligned} 0 < G(e_1, e_1) &= \omega_0(\rho(\eta)e_1, e_1) \\ &= \omega_0(\rho_1(\eta)e_1, e_1) \\ &= \mathrm{Im}(\rho_1(\eta))\omega_0(Ie_1, e_1) \\ &= \mathrm{Im}(\rho_1(\eta))G_0(e_1, e_1), \end{aligned}$$

and hence $\mathrm{Im}(\rho_1(\eta)) > 0$. On the other hand,

$$0 < G(e_2, e_2) = \omega_0(\rho_2(\eta)e_2, e_2) = -\mathrm{Im}\,\rho_1(\eta)G_0(e_2, e_2).$$

A contradiction. It follows that $\Psi$ consists exactly of $n$ copies of a CM-type $\Phi := \{\sigma_1, \ldots, \sigma_m\}$ of $K$.

Now we show that this implies

$$I \in K \otimes_{\mathbb{Q}} \mathbb{R}, \tag{7}$$

or equivalently, $\rho(I) \in \rho(K) \otimes_{\mathbb{Q}} \mathbb{R}$, where $K \otimes_{\mathbb{Q}} \mathbb{R}$ is considered as a subspace of $F \otimes_{\mathbb{Q}} \mathbb{R}$. Since $\rho(I)$ is just the multiplication by $i$ on $\Gamma_{\mathbb{C}}^{1,0}$ and as we showed that $\Psi$ consists of $n$ copies of $\Phi$, (7) amounts to show that there is a unique element $x \in K \otimes_{\mathbb{Q}} \mathbb{R}$, such that for all $\sigma_l \in \Phi$ we have $\sigma_l(x) = i$. This is clear due to the isomorphism:

$$\begin{aligned} K \otimes_{\mathbb{Q}} \mathbb{R} &\xrightarrow{\sim} \mathbb{C}^m \\ a &\longmapsto (\sigma_1(a), \ldots, \sigma_m(a))^t, \end{aligned}$$

where we extended $\sigma_l$ $\mathbb{R}$-linearly.

Finally, as $K$ acts by left multiplication on each copy of $K$ under the isomorphism $\Gamma_{\mathbb{Q}} \cong K^n$ and hence leaves each copy invariant, (7) implies that $I$ leaves each copy of $K \otimes_{\mathbb{Q}} \mathbb{R}$ invariant under the isomorphism $\Gamma_{\mathbb{R}} \cong (K \otimes_{\mathbb{Q}} \mathbb{R})^n$. By the simplicity of $X$ we get $n = 1$. This is what we wanted to show. □

**Remark 2.7.** From the proof above, one sees that $\eta$ can be taken as $\beta$ to define the Riemann form $E$ in (1). Moreover, we could also conclude the proof above by pointing out that (7) means nothing but that the Hodge group $\mathrm{Hg}(X)$ is contained in $K$, which implies immediately that $\mathrm{Hg}(X)$ is commutative and hence $X$ is of CM-type.

## 3. Complex multiplication of the mirror

The aim of this section is to show Theorem 3.11. It gives a sufficient condition which ensures that complex multiplication is transmitted to the mirror partners. Mirror symmetry for Abelian varieties was treated in [4]. Some of their results can be rephrased more naturally in terms of generalized complex structures, which were introduced subsequently by Hitchin in [7] (see also [5,10,1] for further works in generalized complex geometry).

**Definition 3.1.** A generalized complex structure on a smooth manifold $Y$ is a bundle map $\mathcal{I} : TY \oplus T^*Y \longrightarrow TY \oplus T^*Y$ satisfying

(i) $\mathcal{I}^2 = -\mathrm{id}$,
(ii) $\mathcal{I}$ preserves the pseudo-Euclidean metric $q$ on $TY \oplus T^*Y$, where

$$q((a_1, a_2), (b_1, b_2)) := -\langle a_1, b_2\rangle - \langle a_2, b_1\rangle,$$

(iii) $\mathcal{I}$ is integrable with respect to the Courant bracket.

So the $J_{A\times\hat{A}}$ and $I_{\omega_A}$ which appeared in [4] are examples of generalized complex structures, and the couple $(J_{A\times\hat{A}}, I_{\omega_A})$ forms a generalized Kähler structure, which is defined as follows (see [5, Chap. 6] or [12, Section 8]).

**Definition 3.2.** A generalized Kähler structure (GKS) on a smooth manifold $Y$ is a pair $(\mathcal{I}, \mathcal{J})$ of commuting generalized complex structures such that $\mathcal{G}(\cdot,\cdot) := q(\cdot, \mathcal{I}\mathcal{J}\cdot)$ is a positive definite metric on $TY \oplus T^*Y$.

Using GKS, one can define mirror symmetry for a more general class of tori.

**Definition 3.3.** A generalized complex torus $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ is a real torus $\mathbb{T}$ endowed with a GKS $(\mathcal{I}, \mathcal{J})$.

**Definition 3.4.** Two generalized complex tori $(\mathbb{T} = V/\Gamma, \mathcal{I}, \mathcal{J})$ and $(\mathbb{T}' = V'/\Gamma', \mathcal{I}', \mathcal{J}')$ are mirror of each other, if there is a lattice isomorphism

$$\varphi : \Gamma \oplus \Gamma^* \longrightarrow \Gamma' \oplus \Gamma'^*,$$

such that $q(\cdot,\cdot) = q'(\varphi\cdot, \varphi\cdot)$, $\mathcal{I}' = \varphi\mathcal{J}\varphi^{-1}$, and $\mathcal{J}' = \varphi\mathcal{I}\varphi^{-1}$. We call $\varphi$ a mirror map. We also denote by $\varphi$ its $\mathbb{R}$-linear extension.

In Theorem 5.10 we will see that this notion of mirror symmetry between generalized complex tori corresponds exactly to the mirror symmetry between the $N = 2$ lattice vertex algebras associated to them.

Although GKSs provide a more general framework, a special kind of GKS interests us more particularly.

**Definition 3.5.** Let $(T, G, B)$ be a complex torus $T$ with complex structure $I$ and endowed with a constant Kähler metric $G$ and a B-field $B \in H^2(T, \mathbb{R})$. Denote by $\mathbb{T}$ the underlying real torus. Consider the pair $(\mathcal{I}, \mathcal{J})$ defined by

$$\mathcal{I} := \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ BI + I^t B & -I^t \end{pmatrix},$$

$$\begin{aligned} \mathcal{J} : &= \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} = \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix} \\ &= \begin{pmatrix} -IG^{-1}B & IG^{-1} \\ GI - BIG^{-1}B & BIG^{-1} \end{pmatrix}, \end{aligned} \tag{8}$$

where $\omega$ is the Kähler form $G(\cdot, I\cdot)$. We say that the triple $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ as defined above is induced by $(T, G, B)$.

Obviously the above-defined $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ is a generalized complex torus. Moreover, in this case, since $\mathcal{I}$ is the B-transform of the usual complex structure and $\mathcal{J}$ is the B-transform of a symplectic structure, one can say that a mirror map exchanges the complex structure and the symplectic structure of the mirror partners. Thus, the language of GKS provides a conceptually clean approach to mirror symmetry. We formulate this more precisely in the following

**Definition 3.6.** We say that two complex tori $(T, G, B)$ and $(T', G', B')$ with complex structure $I$ respectively $I'$, a constant Kähler metric $G$ respectively $G'$ and a B-field $B$ respectively $B'$ are mirror partners, if the generalized complex tori they induce as in Definition 3.5 are mirror of each other.

In order to prove Theorem 3.11 we give the next lemma which first studies the rationality of the composition $\mathcal{I}\mathcal{J}$ on a generalized complex torus $(\mathbb{T}, \mathcal{I}, \mathcal{J})$, then links the rationality of $G$ and $B$ to the rationality of $\mathcal{I}\mathcal{J}$ of the GKS they induce. The rationality of $\mathcal{I}\mathcal{J}$ is defined as follows.

**Definition 3.7.** Let $(\mathbb{T} = V/\Gamma, \mathcal{I}, \mathcal{J})$ be a generalized complex torus. Denote

$$\Lambda := \Gamma \oplus \Gamma^*.$$

We identify the tangent space of $\mathbb{T}$ with $\Gamma_{\mathbb{R}}$. We say that the composition $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$ if $\mathcal{I}\mathcal{J}$ preserves the rational lattices:

$$\mathcal{I}\mathcal{J} : \Lambda_{\mathbb{Q}} \longrightarrow \Lambda_{\mathbb{Q}}.$$

**Lemma 3.8.** *Let $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ be a generalized complex torus. Denote by $C_\pm \subset \Lambda_{\mathbb{R}}$ the eigenspace of $\mathcal{I}\mathcal{J}$ with eigenvalue $\pm 1$. Then*

(i) *We have*

$$C_\pm = (\pm \operatorname{Id} + \mathcal{I}\mathcal{J})(\Lambda_{\mathbb{R}}) = \operatorname{Image}_{\Lambda_{\mathbb{R}}}(\pm \operatorname{Id} + \mathcal{I}\mathcal{J}),$$

*and we have the decomposition*

$$\Lambda_{\mathbb{R}} = C_+ \oplus C_-. \tag{9}$$

*This decomposition is orthogonal with respect to $q$ and it is defined over $\mathbb{Q}$ if and only if $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$. Moreover, $q$ is positive definite on $C_+$ and negative definite on $C_-$.*

(ii) *If $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ is induced by $(T, G, B)$, then we have*

$$C_\pm = \operatorname{Graph}_{\Gamma_{\mathbb{R}}}(\mp G + B).$$

*Moreover, the following is equivalent:*
- *The decomposition (9) is defined over $\mathbb{Q}$,*
- *$\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$,*
- *$G$ and $B$ are both rational.*

**Proof.** The orthogonality of (9) and the definiteness of $q$ on $C_\pm$ follows from $\mathcal{I}, \mathcal{J} \in O(q)$ and the requirement that $\mathcal{G} = q\mathcal{I}\mathcal{J}$ is positive definite. The rest of (i) is due to the commutativity $\mathcal{I}\mathcal{J} = \mathcal{J}\mathcal{I}$. To prove (ii) it suffices to note that

$$\mathcal{I}\mathcal{J}\begin{pmatrix} 1 \\ \mp G + B \end{pmatrix} = \pm \begin{pmatrix} 1 \\ \mp G + B \end{pmatrix}. \quad \square$$

The following lemma shows how the lattice of mirror partners is related to each other.

**Lemma 3.9.** *Let $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ and $(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ be mirror generalized complex tori and $\varphi$ a mirror map between them. Then*

(i) *$\varphi$ respects the decomposition (9), i.e. $\varphi : C_\pm \to C'_\pm$. In particular, $C_+ \oplus C_-$ is defined over $\mathbb{Q}$ if and only if $C'_+ \oplus C'_-$ is defined over $\mathbb{Q}$.*

(ii) *If* $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ *and* $(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ *are induced by* $(T, G, B)$ *respectively* $(T', G', B')$*, then* $\varphi$ *induces isomorphisms* $\psi_\pm : \Gamma_\mathbb{R} \xrightarrow{\sim} \Gamma'_\mathbb{R}$ *defined by*

$$\varphi(a, (\mp G + B)a) = (\psi_\pm a, (\mp G' + B')\psi_\pm a), \tag{10}$$

*and we have*
(a) $G(a, b) = G'(\psi_\pm a, \psi_\pm b)$ *for all* $a, b \in \Gamma_\mathbb{R}$.
(b) $I' = \psi_+ \circ I \circ \psi_+^{-1} = \psi_- \circ I \circ \psi_-^{-1}$.

**Proof.** (i) is immediate. For (ii)(a) we make an explicit calculation

$$q\,((a, (\mp G + B)a), (b, (\mp G + B)b)) = \pm 2G(a, b) \quad \forall a, b \in \Gamma_\mathbb{R}.$$

Then use $q(\cdot, \cdot) = q'(\varphi\cdot, \varphi\cdot)$ to get the claim. For (ii)(b) we verify the equality for $\psi_-$, the case of $\psi_+$ is similar. Recalling the definition of the generalized Kähler structure from (8) we have for any $(a', (G' + B')a') \in C'_-$:

$$\mathcal{I}' \begin{pmatrix} a' \\ (G' + B')a' \end{pmatrix} = \begin{pmatrix} I'a' \\ * \end{pmatrix}, \tag{11}$$

we are only interested in the first component. Using (10) the left hand side of (11) is

$$\varphi \mathcal{J} \varphi^{-1} \begin{pmatrix} a' \\ (G' + B')a' \end{pmatrix} = \begin{pmatrix} \psi_- I \psi_-^{-1} a' \\ (G + B)\psi_- I \psi_-^{-1} a' \end{pmatrix}.$$

Comparing with the right hand side of (11), we obtain (ii)(b). □

As mirror symmetry exchanges complex and Kähler structures, two mirror Calabi–Yau manifolds are in general very different as complex manifolds. This remains true for Abelian varieties, but surprisingly the following result shows that it suffices that the composition $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$ for the mirror to be an isogenous complex torus.

**Proposition 3.10.** *Let* $(T, G, B)$ *and* $(T', G', B')$ *be mirror partners. If* $G$ *and* $B$ *are both rational, then* $T$ *and* $T'$ *are isogenous.*

**Proof.** By Lemma 3.8(ii), $G$ and $B$ are both rational if and only if $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$. Then Lemma 3.9 (i) implies that $G'$ and $B'$ are rational. By (10) $\psi_\pm$ are then defined over $\mathbb{Q}$. Finally, from (ii)(b) of the same lemma, it follows that some integral multiple of $\psi_\pm$ is actually an isogeny between $T$ and $T'$. □

This immediately implies the following

**Theorem 3.11.** *Let* $(X, G, B)$ *and* $(X', G', B')$ *be mirror Abelian varieties. Suppose that* $X$ *is of CM-type. If both* $G$ *and* $B$ *are rational, then* $X$ *and* $X'$ *are isogenous. In particular,* $X'$ *is also of CM-type.*

In the next section we will show that the converse of Theorem 3.11 however does not hold. This will also have some consequence in terms of vertex algebras (see Corollary 5.13).

## 4. An example of mirror Abelian varieties of CM-type

In this section we show that the converse of Theorem 3.11 does not hold.

**Proposition 4.1.** *There are mirror Abelian varieties* $(X, G, B)$ *and* $(X', G', B')$*, such that* $X$ *and* $X'$ *are isogenous and of CM-type, but neither* $\mathcal{I}\mathcal{J}$ *nor* $\mathcal{I}'\mathcal{J}'$ *is defined over* $\mathbb{Q}$*, where* $(\mathcal{I}, \mathcal{J})$ *and* $(\mathcal{I}', \mathcal{J}')$ *denote their induced GKS.*

We shall eventually construct an explicit example of such mirror pairs, but we need first some preparation.

**Lemma 4.2.** *Let* $T \cong \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$ *and* $T' \cong \mathbb{C}^g/\Pi'\mathbb{Z}^{2g}$ *be complex tori with period matrix* $\Pi$ *and* $\Pi'$ *respectively. Then* $T$ *and* $T'$ *are isogenous if and only if there is a complex matrix* $C \in GL(g, \mathbb{C})$ *and a rational matrix* $\gamma \in GL(2g, \mathbb{Q})$*, such that*

$$\Pi' = C\Pi\gamma. \tag{12}$$

*In particular, if there is a such matrix* $\gamma$*, then* $I' = \gamma^{-1} I \gamma$ *and if* $G$ *is a Kähler metric on* $T$ *then* $G'(\cdot, \cdot) := G(\gamma\cdot, \gamma\cdot)$ *is a Kähler metric on* $T'$.

**Proof.** It is easy to see that the rational respectively analytic representation of any isogeny provides the matrix $\gamma$ respectively $C$. For the converse, recall that in general, the rational representation of $I$ of a complex torus with period matrix $\Pi$ is $I = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}^{-1} \begin{pmatrix} i\mathbf{1} & 0 \\ 0 & -i\mathbf{1} \end{pmatrix} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$. Replacing $\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$ by $\begin{pmatrix} \Pi' \\ \overline{\Pi'} \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} \gamma$ for $I'$, we get $I' = \gamma^{-1} I \gamma$. It follows that some integral multiple of $\gamma$ is an isogeny. The rest of the claim is obvious. □

Next we give a special form (see (13) below) of the period matrix, which makes the construction of an isogenous mirror easier. Later we will give an Abelian variety of CM-type over a cyclotomic field, whose period matrix can be written in the form (13). First a lemma which expresses $I$ explicitly.

**Lemma 4.3.** *Let $\Gamma$ be the lattice of a complex torus $T$ generated by $e_1, \ldots, e_{2g}$. Suppose that the period matrix $\Pi$ of $T$ in the complex basis $\{e_1, \ldots, e_g\}$ has the form $\Pi = (\mathbf{1}\ T_1 + T_2 i)$, where $T_1$ and $T_2$ are real $g \times g$ matrices, then in the basis $\{e_1, \ldots, e_{2g}\}$ we have*

$$I = \begin{pmatrix} -T_1 T_2^{-1} & -T_1 T_2^{-1} T_1 - T_2 \\ T_2^{-1} & T_2^{-1} T_1 \end{pmatrix}.$$

**Proof.** The proof is a matter of calculating the matrix $I = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}^{-1} \begin{pmatrix} i\mathbf{1} & 0 \\ 0 & -i\mathbf{1} \end{pmatrix} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$, where

$$\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}^{-1} = \frac{i}{2} \begin{pmatrix} T_1 T_2^{-1} - i & -T_1 T_2^{-1} - i \\ -T_2^{-1} & T_2^{-1} \end{pmatrix}. \quad \square$$

**Proposition 4.4.** *If an Abelian variety $X$ has a period matrix of the form*

$$\Pi = \begin{pmatrix} \mathbf{1} & Ai \end{pmatrix}, \tag{13}$$

*with a real matrix $A \in GL(g, \mathbb{R})$, then by choosing a suitable constant Kähler metric $G$ and by setting $B = 0$, one can find an isogenous mirror Abelian variety $(X', G', B')$.*

**Proof.** Suppose that $\Pi$ has the form in (13). Then by Lemma 4.3 we have $I = \begin{pmatrix} 0 & -A \\ A^{-1} & 0 \end{pmatrix}$. Let us choose the metric

$$G = \begin{pmatrix} -\rho & 0 \\ 0 & -A^t \rho A \end{pmatrix},$$

where $\rho$ is a symmetric negative definite matrix with integral coefficients. One verifies that $G$ is compatible with $I$, i.e. $I^t G I = G$, so $G$ is a Kähler metric. Setting $B = 0$, then by (8) $\mathcal{I}$ and $\mathcal{J}$ have the form

$$\mathcal{I} = \begin{pmatrix} 0 & -A & 0 & 0 \\ A^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -A^{-1t} \\ 0 & 0 & A^t & 0 \end{pmatrix}, \qquad \mathcal{J} = \begin{pmatrix} 0 & 0 & 0 & \rho^{-1} A^{-1t} \\ 0 & 0 & -A^{-1}\rho^{-1} & 0 \\ 0 & \rho A & 0 & 0 \\ -A^t \rho & 0 & 0 & 0 \end{pmatrix}.$$

Further, we choose

$$C = \rho \quad \text{and} \quad \gamma = \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

to get a new period matrix

$$\Pi' = C \Pi \gamma = \begin{pmatrix} \mathbf{1} & \rho A i \end{pmatrix}.$$

Then by Lemma 4.2 the complex torus $X' := \mathbb{C}^g / \Pi' \mathbb{Z}^{2g}$ has complex structure respectively Kähler metric

$$I' = \gamma^{-1} I \gamma = \begin{pmatrix} 0 & -\rho A \\ A^{-1} \rho^{-1} & 0 \end{pmatrix} \quad \text{resp. } G' = \gamma^t G \gamma = \begin{pmatrix} -\rho^{-1} & 0 \\ 0 & -A^t \rho A \end{pmatrix}.$$

Setting $B' = 0$ we get

$$\mathcal{I}' = \begin{pmatrix} 0 & -\rho A & 0 & 0 \\ A^{-1}\rho^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\rho^{-1}A^{-1t} \\ 0 & 0 & A^t\rho & 0 \end{pmatrix}, \qquad \mathcal{J}' = \begin{pmatrix} 0 & 0 & 0 & A^{-1t} \\ 0 & 0 & -A^{-1} & 0 \\ 0 & A & 0 & 0 \\ -A^t & 0 & 0 & 0 \end{pmatrix}.$$

By easy calculations, one verifies that $(X, G, B)$ and $(X', G', B')$ as defined above are mirror partners with the mirror map

$$\varphi = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & -\mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{1} \end{pmatrix} : \Gamma \oplus \Gamma^* \longrightarrow \Gamma' \oplus \Gamma'^*.$$

Hence, to any Abelian variety with period matrix $\Pi = (\mathbf{1}\ \ Ai)$ by choosing a suitable $G$ and B-field one can find an isogenous mirror Abelian variety. $\square$

In order to obtain a mirror pair of CM-type, let us consider the cyclotomic field $K = \mathbb{Q}(\xi)$ with $\xi^5 = 1, \xi \neq 1$, which is a CM-field. Denote by

$$w := \mathrm{e}^{\frac{2\pi}{5}i} = \frac{1}{4}(-1+\sqrt{5}) + \frac{i}{2}\sqrt{\frac{1}{2}(5+\sqrt{5})},$$

one can write the four embeddings of $K$ into $\mathbb{C}$ as

$$\sigma_k : \xi \longmapsto w^k, \quad k = 1, \dots, 4.$$

One has $\sigma_1 = \bar{\sigma}_4$ and $\sigma_2 = \bar{\sigma}_3$. Choose the CM-type $\Phi = \{\sigma_1, \sigma_2\}$, and choose the lattice to be the ring of integers

$$\Gamma = \mathcal{O}_K = \mathbb{Z}[\xi].$$

The complex torus $X := \mathbb{C}^2/\Phi(\mathcal{O}_K)$ is then an Abelian variety of CM-type over $K$ (see Section 2). Let us fix the following generators for $\Gamma$:

$$\Gamma = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}(\xi+\xi^{-1}) \oplus \mathbb{Z}(\xi-\xi^{-1}) \oplus \mathbb{Z}(\xi^2-\xi^{-2}).$$

Then under $\Phi$ the lattice is

$$\Phi(\mathcal{O}_K) = \begin{pmatrix} 1 & w+w^{-1} & w-w^{-1} & w^2-w^{-2} \\ 1 & w^2+w^{-2} & w^2-w^{-2} & w^4-w^{-4} \end{pmatrix}\mathbb{Z} =: \begin{pmatrix} Z & Ai \end{pmatrix}\mathbb{Z}.$$

The left two columns form a real matrix $Z$, while the right two columns form a purely imaginary matrix which we write as $Ai$ where $A$ is a real matrix. Choosing the first two generators to be a complex basis of $\Phi(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{R}$, we get the period matrix

$$\Pi = \begin{pmatrix} \mathbf{1} & Z^{-1}Ai \end{pmatrix}$$

in the form (13) with the real matrix $Z^{-1}A$. Together with

$$G = \begin{pmatrix} -\rho & 0 \\ 0 & -A^tZ^{-1t}\rho Z^{-1}A \end{pmatrix} \quad \text{and} \quad B = 0,$$

the Abelian variety $(X, G, B)$ possesses an isogenous mirror $(X', G', B')$ as constructed in Proposition 4.4.

The last step to get $(\mathcal{I}, \mathcal{J})$ such that the composition $\mathcal{I}\mathcal{J}$ is not defined over $\mathbb{Q}$ is to choose an appropriate $\rho$. As claimed by Lemma 3.8(ii), $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$ if and only if $G$ and $B$ are both rational. We set $B = 0$ and choose

$$\rho = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},$$

then

$$G = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 + \frac{2}{\sqrt{5}} & 2 - \frac{1}{\sqrt{5}} \\ 0 & 0 & 2 - \frac{1}{\sqrt{5}} & 3 - \frac{2}{\sqrt{5}} \end{pmatrix},$$

which is not rational. Hence $\mathcal{I}\mathcal{J}$ is not defined over $\mathbb{Q}$, although $X$ is of CM-type and has a mirror $(X', G', B')$ of CM-type over the same field $K$. This shows Proposition 4.1.

## 5. Rationality of lattice vertex algebras, mirror symmetry and complex multiplication

We construct in Section 5.1 a vertex algebra $V(\Lambda, q, \Lambda_z)$ in the sense of [13]. In the special case where $\Lambda_z = \Lambda_{\mathbb{R}}$ (see (14) below), $V(\Lambda, q, \Lambda_z)$ shall be the classical lattice vertex algebra described in [11, Section 5.4]. We will, therefore, call $V(\Lambda, q, \Lambda_z)$ a lattice vertex algebra also in the general case. In Section 5.2 we will see how a generalized complex torus $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ gives rise to such a lattice vertex algebra. In the Appendix we show that it coincides with the toroidal vertex algebra in [13]. In Section 5.3 we discuss the notion of rationality (the precise meaning of this will be recalled). In Section 5.4 we rephrase results of the preceding sections in terms of lattice vertex algebras, and answer the question (Q) posed in the introduction.

### 5.1. Construction of lattice vertex algebra $V(\Lambda, q, \Lambda_z)$

We begin with an integral lattice $(\Lambda, q)$, together with a $(z, \bar{z})$-decomposition

$$\Lambda_{\mathbb{R}} = \Lambda_z \oplus \Lambda_{\bar{z}} \tag{14}$$

over $\mathbb{R}$, which is orthogonal with respect to $q$. This shall suffice to construct an $N = 1$ vertex algebra. If moreover we endow the vector space $\Lambda_{\mathbb{R}}$ with an almost complex structure $\mathcal{I}$, i.e. $\mathcal{I}^2 = -\mathrm{Id}$, then we will get an $N = 2$ vertex algebra (see [13, Section 3] for definitions). We shall write $a = a_z - a_{\bar{z}}$ with $a_z \in \Lambda_z$ and $a_{\bar{z}} \in \Lambda_{\bar{z}}$. Introduce two copies $\mathfrak{h}_b = \mathfrak{h}_f = \Lambda_{\mathbb{C}}$ of $\Lambda_{\mathbb{C}}$, which both inherit the decomposition

$$\mathfrak{h}_b = \mathfrak{h}_{bz} \oplus \mathfrak{h}_{b\bar{z}} \quad \text{and} \quad \mathfrak{h}_f = \mathfrak{h}_{fz} \oplus \mathfrak{h}_{f\bar{z}}.$$

The affinization is the Lie superalgebra

$$\hat{\mathfrak{h}} := \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h}_b \oplus t^{\frac{1}{2}}\mathbb{C}[t, t^{-1}] \otimes \mathfrak{h}_f \oplus \mathbb{C}K$$

with even $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{h}_b \oplus \mathbb{C}K$, and odd $t^{\frac{1}{2}}\mathbb{C}[t, t^{-1}] \otimes \mathfrak{h}_f$. The supercommutators are described below by (15). Write $\mathbb{C}[\Lambda]$ for the group algebra of $\Lambda$ over $\mathbb{C}$, and denote by $e^a$, $a \in \Lambda$ the basis vectors of $\mathbb{C}[\Lambda]$. Furthermore, write $\mathfrak{h}_b^< := t^{-1}\mathfrak{h}_b[t^{-1}]$, and $\mathfrak{h}_f^< := t^{-\frac{1}{2}}\mathfrak{h}_f[t^{-1}]$. The space of states is the superspace

$$V := \operatorname{Sym} \mathfrak{h}_b^< \otimes \bigwedge \mathfrak{h}_f^< \otimes \mathbb{C}[\Lambda],$$

where $\operatorname{Sym} \mathfrak{h}_b^<$ is the symmetric algebra of $\mathfrak{h}_b^<$ and $\bigwedge \mathfrak{h}_f^<$ is the exterior algebra of $\mathfrak{h}_f^<$. The vacuum vector $\mathbf{1} = |vac\rangle := 1 \otimes 1 \otimes 1$. The parity on $V$ is $p(s \otimes e^a) = q(a, a) \bmod 2 + p(s)$. The representation $\pi$ of $\hat{\mathfrak{h}}$ on $V$ is defined as

$$\pi := \pi_1 \otimes 1 + 1 \otimes \pi_2,$$

with $\pi_1$ the representation of $\hat{\mathfrak{h}}$ on $\operatorname{Sym} \mathfrak{h}_b^< \otimes \bigwedge \mathfrak{h}_f^<$ determined by

$$\begin{aligned}
&K \longmapsto \mathrm{Id},\\
&h \longmapsto 0, \quad h \in \mathfrak{h}_b,\\
&n < 0, \quad t^n h \longmapsto \text{ multiplication by } t^n h,\, h \in \mathfrak{h}_b \text{ or } h \in \mathfrak{h}_f,\\
&n > 0, \quad t^n h \longmapsto (t^{-s}h' \mapsto n\delta_{n,s}q(h, h')),\, h, h' \in \mathfrak{h}_{bz},
\end{aligned}$$

$$t^n\bar{h} \longmapsto (t^{-s}\bar{h}' \mapsto -n\delta_{n,s}q(\bar{h},\bar{h}')), \quad \bar{h},\bar{h}' \in \mathfrak{h}_{b\bar{z}},$$
$$t^n h \longmapsto (t^{-s}h' \mapsto \delta_{n,s}q(h,h')), \quad h,h' \in \mathfrak{h}_{fz},$$
$$t^n\bar{h} \longmapsto (t^{-s}\bar{h}' \mapsto -\delta_{n,s}q(\bar{h},\bar{h}')), \quad \bar{h},\bar{h}' \in \mathfrak{h}_{f\bar{z}},$$

and $\pi_2$ the representation of $\hat{\mathfrak{h}}$ on $\mathbb{C}[\Lambda]$ determined by:

$$K \longmapsto 0,$$
$$t^n h \longmapsto (e^a \mapsto \delta_{n,0}q(h,a)e^a), \quad h \in \mathfrak{h}_{bz},$$
$$t^n\bar{h} \longmapsto (e^a \mapsto -\delta_{n,0}q(\bar{h},a)e^a), \quad \bar{h} \in \mathfrak{h}_{b\bar{z}},$$
$$t^n h \longmapsto 0, \quad h \in \mathfrak{h}_f, \forall n.$$

If we write $h_n := \pi(t^n h)$, then for $h \in \mathfrak{h}_{bz}$, $\bar{h} \in \mathfrak{h}_{b\bar{z}}$, $f \in \mathfrak{h}_{fz}$, $\bar{f} \in \mathfrak{h}_{f\bar{z}}$, and $m,n \in \mathbb{Z}$, $r,s \in \mathbb{Z}+\frac{1}{2}$ the supercommutators are

$$\begin{aligned} &[h_n,h'_m] = n\delta_{n,-m}q(h,h'), \qquad [\bar{h}_n,\bar{h}'_m] = -n\delta_{n,-m}q(\bar{h},\bar{h}'),\\ &\{f_r,f'_s\} = \delta_{r,-s}q(f,f'), \qquad \{\bar{f}_r,\bar{f}'_s\} = -\delta_{r,-s}q(\bar{f},\bar{f}'), \end{aligned} \tag{15}$$

and all other relations are trivial. The state-field correspondence maps a homogeneous vector

$$v = h^1_{-s_1}\cdots h^n_{-s_n}\bar{h}^1_{-\bar{s}_1}\cdots\bar{h}^{\bar{n}}_{-\bar{s}_{\bar{n}}}f^1_{-r_1}\cdots f^q_{-r_q}\bar{f}^1_{-\bar{r}_1}\cdots\bar{f}^{\bar{q}}_{-\bar{r}_{\bar{q}}}\otimes \mathrm{e}^a, \tag{16}$$

where $s_i, \bar{s}_{\bar{i}}$ are positive integers and $r_i, \bar{r}_{\bar{i}}$ are positive half-integers, to the field

$$\begin{aligned} v(z,\bar{z}) := Y(v,z,\bar{z}) = \sum_{b\in\Lambda}\epsilon(a,b)e^a\mathrm{Pr}_b z^{q(a_z,b_z)}\bar{z}^{-q(a_{\bar{z}},b_{\bar{z}})}\exp\left(-\sum_{n<0}\frac{a_{zn}}{nz^n}+\sum_{n<0}\frac{a_{\bar{z}n}}{n\bar{z}^n}\right)\\ \times :\prod_{l=1}^{n}\frac{\partial^{s_l}H^l(z)}{(s_l-1)!}\prod_{\bar{l}=1}^{\bar{n}}\frac{\bar{\partial}^{\bar{s}_{\bar{l}}}\bar{H}^{\bar{l}}(\bar{z})}{(\bar{s}_{\bar{l}}-1)!}\prod_{t=1}^{q}\frac{\partial^{r_t-\frac{1}{2}}F^t(z)}{(r_t-\frac{1}{2})!}\prod_{\bar{t}=1}^{\bar{q}}\frac{\bar{\partial}^{\bar{r}_{\bar{t}}-\frac{1}{2}}\bar{F}^{\bar{t}}(\bar{z})}{(\bar{r}_{\bar{t}}-\frac{1}{2})!}:\\ \times\exp\left(-\sum_{n>0}\frac{a_{zn}}{nz^n}+\sum_{n>0}\frac{a_{\bar{z}n}}{n\bar{z}^n}\right), \end{aligned} \tag{17}$$

where $\mathrm{Pr}_b$ is the projection onto $\mathrm{Sym}\,\mathfrak{h}_b^< \otimes \bigwedge \mathfrak{h}_f^< \otimes e^b$ and

$$\partial H(z) := \sum_{m\in\mathbb{Z}} h_m z^{-m-1}, \qquad \bar{\partial}\bar{H}(\bar{z}) := \sum_{m\in\mathbb{Z}}\bar{h}_m\bar{z}^{-m-1},$$
$$F(z) := \sum_{r\in\mathbb{Z}+\frac{1}{2}} f_r z^{-r-\frac{1}{2}}, \qquad \bar{F}(\bar{z}) := \sum_{r\in\mathbb{Z}+\frac{1}{2}}\bar{f}_r\bar{z}^{-r-\frac{1}{2}},$$

and the factor $\epsilon(a,b)$ satisfies the equations (5.4.14) in [11]. We give a few examples:

- $v = 1\otimes e^0 = \mathbf{1}$, $Y(v,z,\bar{z}) = \sum_{b\in\Lambda}\mathrm{Pr}_b = id$,
- $v = h_{-s}\otimes e^0$, $Y(v,z,\bar{z}) = \frac{1}{(s-1)!}\partial^s H(z)$,
- $v = f_{-r}\otimes e^0$, $Y(v,z,\bar{z}) = \frac{1}{(r-\frac{1}{2})!}\partial^{r-\frac{1}{2}}F(z)$,
- $v = 1\otimes e^a$,

$$\begin{aligned} Y(v,z,\bar{z}) = &\sum_{b\in\Lambda}\epsilon(a,b)e^a\mathrm{Pr}_b z^{q(a_z,b_z)}\bar{z}^{-q(a_{\bar{z}},b_{\bar{z}})}\\ &\times\exp\left(-\sum_{n<0}\frac{a_{zn}}{nz^n}+\sum_{n<0}\frac{a_{\bar{z}n}}{n\bar{z}^n}\right)\exp\left(-\sum_{n>0}\frac{a_{zn}}{nz^n}+\sum_{n>0}\frac{a_{\bar{z}n}}{n\bar{z}^n}\right). \end{aligned}$$

Next we define the maps $T$ and $\bar{T}$. Let $\{E^i\}$ respectively $\{\bar{E}^i\}$ be a bosonic basis of $\Lambda_z$ respectively $\Lambda_{\bar{z}}$ and $\{\widetilde{E}^i\}$ respectively $\{\widetilde{\bar{E}}^i\}$ be the dual basis with respect to $q$, i.e. $q(E^i, \widetilde{E}^j) = \delta^{ij}$ and $q(\bar{E}^i, \widetilde{\bar{E}}^j) = -\delta^{ij}$. The fermionic bases are denoted by $\{F^i\}$, $\{\bar{F}^i\}$ and $\{\widetilde{F}^i\}$, $\{\widetilde{\bar{F}}^i\}$. Then

$$T := \sum_i \left( \sum_{n \geq 0} E^i_{-n-1} \widetilde{E}^i_n + \sum_{r = \frac{1}{2}, \frac{3}{2} \dots} (r + \frac{1}{2}) F^i_{-r-1} \widetilde{F}^i_r \right), \quad \bar{T} \text{ is analogous.}$$

The superconformal structure is:

$$L := \frac{1}{2} \sum_i (E^i_{-1} \widetilde{E}^i_{-1} - F^i_{-\frac{1}{2}} \widetilde{F}^i_{-\frac{3}{2}}) \otimes e^0, \quad \bar{L} \text{ is analogous.}$$

**Remark 5.1.** By inspecting the state-field correspondence (17) one sees that fields may have non-integral powers in $z$ and $\bar{z}$ due to the term $z^{q(a_z, b_z)} \bar{z}^{-q(a_{\bar{z}}, b_{\bar{z}})}$ (recall that the $(z, \bar{z})$-decomposition (14) is only required to be defined over $\mathbb{R}$). However, the difference of the exponents of $z$ and $\bar{z}$ is always an integer:

$$q(a_z, b_z) + q(a_{\bar{z}}, b_{\bar{z}}) = q(a, b) \in \mathbb{Z}.$$

This implies that in the special case of a trivial decomposition, i.e. $\Lambda_{\bar{z}} = 0$ or in other words $\Lambda_{\mathbb{R}} = \Lambda_z$, the lattice vertex algebra is nothing but a conformal lattice vertex algebra in the sense of [11, Section 5.4 Section 5.5] (plus a fermionic part).

The $N = 1$ structure is

$$Q := \frac{i}{2\sqrt{2}} \sum_i F^i_{-\frac{1}{2}} \widetilde{E}^i_{-1} \otimes e^0.$$

Given an almost complex structure $\mathcal{I}$ on the vector space $\Lambda_{\mathbb{R}}$, the $N = 2$ structure is denoted by

$$Q^{\pm} := \frac{i}{4\sqrt{2}} \sum_i (F^i_{-\frac{1}{2}} \widetilde{E}^i_{-1} \pm F^i_{-\frac{1}{2}} \mathcal{I} \widetilde{E}^i_{-1}) \otimes e^0,$$

$$J := -\frac{i}{2} \sum_i F^i_{-\frac{1}{2}} \mathcal{I} \widetilde{F}^i_{-\frac{1}{2}} \otimes e^0.$$

and the analogous $\bar{z}$-part.

### 5.2. Toroidal lattice vertex algebras

Now we explain how tori give rise to lattice vertex algebras. To a real torus $\mathbb{T}$ together with a constant metric $G$ and a B-field, one can associate a $N = 1$ superconformal lattice vertex algebra $V(\mathbb{T}, G, B)$ by setting

$$\begin{aligned} &\Lambda = \Gamma \oplus \Gamma^*, \qquad q((a_1, a_2), (b_1, b_2)) := -\langle a_1, b_2 \rangle - \langle a_2, b_1 \rangle \\ &\text{and } \Lambda_z = \mathrm{Graph}_{\Gamma_{\mathbb{R}}}(-G + B), \qquad \Lambda_{\bar{z}} = \mathrm{Graph}_{\Gamma_{\mathbb{R}}}(G + B) \end{aligned} \tag{18}$$

and define the factor $\epsilon(a, b) := \exp(i\pi \langle a_1, b_2 \rangle)$ for $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\Gamma \oplus \Gamma^*$.

**Remark 5.2.** Calculations show that $V(\mathbb{T}, G, B)$ is a theory with central charge $c = \bar{c} = 3d$, where $\mathbb{T}$ has real dimension $2d$. From the expansion of $L$, we get

$$L_0 = \sum_{i=1}^{2d} \left( \frac{1}{2} E^i_0 \tilde{E}^i_0 + \sum_{n \geq 1} E^i_{-n} \tilde{E}^i_n + \sum_{r = \frac{1}{2}, \frac{3}{2}, \dots} r F^i_{-r} \tilde{F}^i_r \right)$$

(analogously for $\bar{L}_0$). The partition function is then

$$\begin{aligned} Z &= \mathrm{Tr}_V\, \mathfrak{q}^{L_0-\frac{c}{24}} \bar{\mathfrak{q}}^{\bar{L}_0-\frac{\bar{c}}{24}} \\ &= \frac{1}{\eta(\tau)^{2d}\eta(\bar{\tau})^{2d}} \left(\frac{\theta_3(\tau)}{\eta(\tau)}\right)^d \left(\frac{\theta_3(\bar{\tau})}{\eta(\bar{\tau})}\right)^d \left(\sum_{a\in\Lambda} \mathfrak{q}^{\frac{1}{2}q(a_z,a_z)} \bar{\mathfrak{q}}^{\frac{1}{2}q(a_{\bar{z}},a_{\bar{z}})}\right), \end{aligned} \tag{19}$$

where we set $\mathfrak{q} = \mathrm{e}^{2\pi\mathrm{i}\tau}$ and $\bar{\mathfrak{q}} = \mathrm{e}^{2\pi\mathrm{i}\bar{\tau}}$. The two modular forms are Jacobi theta function $\theta_3$ and the Dedekind eta function $\eta$. The sum in the last term reveals whether the lattice vertex algebra is rational, see Remark 5.8.

To a generalized complex torus $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ one can associate two $N = 2$ structures. Set $(\Lambda, q)$ and $\epsilon(a, b)$ as in (18) and

$$\Lambda_z = \mathrm{Image}_{\Lambda_{\mathbb{R}}}(\mathrm{Id} + \mathcal{I}\mathcal{J}), \qquad \Lambda_{\bar{z}} = \mathrm{Image}_{\Lambda_{\mathbb{R}}}(-\mathrm{Id} + \mathcal{I}\mathcal{J})$$

(see Lemma 3.8(i)). Now one can choose either $\mathcal{I}$ or $\mathcal{J}$ to define $Q^{\pm}$ and $J$. As $\mathcal{I} = \mathcal{J}$ on $\Lambda_z$ and $\mathcal{I} = -\mathcal{J}$ on $\Lambda_{\bar{z}}$ the two $N = 2$ structures are related as follows:

$$Q^{\pm}_{\mathcal{I}} = Q^{\pm}_{\mathcal{J}}, \qquad \bar{Q}^{\pm}_{\mathcal{I}} = \bar{Q}^{\mp}_{\mathcal{J}}, \qquad J_{\mathcal{I}} = J_{\mathcal{J}}, \quad \bar{J}_{\mathcal{I}} = -\bar{J}_{\mathcal{J}}. \tag{20}$$

For simplicity, we denote by $V(\mathbb{T}, \mathcal{I}, \mathcal{J})$ either the $N = 2$ superconformal lattice vertex algebra defined by $\mathcal{I}$ or $\mathcal{J}$. If additionally, $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ is induced by $(T, G, B)$ (see Definition 3.5), we also write $V(T, G, B)$ for the $N = 2$ lattice vertex algebra $V(\mathbb{T}, \mathcal{I}, \mathcal{J})$.

In the Appendix we prove

**Proposition 5.3.** *The $N = 2$ superconformal lattice vertex algebra $V(T, G, B)$ is isomorphic to the $N = 2$ superconformal vertex algebra constructed in* [13].

Adopting the viewpoint of lattice vertex algebras has the advantage of having a basis-free construction. Moreover, it is more apparent how the lattice determines the structure of the vertex algebra. This leads us to the formulation of rationality for lattice vertex algebras (see Definition 5.5) and eventually to prove that the rationality is completely determined by the size of the so-called chiral sublattice $\Lambda_{ch}$ of $\Lambda$ (see Proposition 5.6).

### *5.3. Rationality*

We phrase the rationality of the lattice vertex algebra $V(\Lambda, q, \Lambda_z)$ in terms of its so-called chiral subalgebra whose fields contain only integral powers in $z$ and $\bar{z}$ (see Definition 5.5). Roughly speaking, we call $V(\Lambda, q, \Lambda_z)$ rational if its chiral subalgebra is so big that $V(\Lambda, q, \Lambda_z)$ breaks into a finite direct sum of irreducible representations of its chiral subalgebra.

As mentioned in Remark 5.1, the term $z^{q(a_z,b_z)}\bar{z}^{-q(a_{\bar{z}},b_{\bar{z}})}$ may give non-integral powers in $z$ and $\bar{z}$. However, the subspace $W \subset V$ of those states which only yield integral powers is easy to describe. If we write

$$\Lambda_{ch} := \{\lambda \in \Lambda \mid q(\lambda_z, \alpha) \in \mathbb{Z}, \forall \alpha \in \Lambda\},$$

then we have

$$W = \mathrm{Sym}\, \mathfrak{h}_b^{<} \otimes \bigwedge \mathfrak{h}_f^{<} \otimes \mathbb{C}[\Lambda_{ch}].$$

We call $\Lambda_{ch}$ the *chiral sublattice* of $\Lambda$. In general, $W$ does not have the structure of a lattice vertex algebra, as $\Lambda_{ch}$ may be of smaller rank than $\Lambda$. However, if we require

(a) $(\Lambda, q)$ is unimodular, i.e. $\Lambda$ is its own dual with respect to $q$, and
(b) $q$ restricted on $\Lambda_{ch}$ is non-degenerate,

then the space

$$V_{ch} := \mathrm{Sym}\, \mathfrak{h}_{ch,b}^{<} \otimes \bigwedge \mathfrak{h}_{ch,f}^{<} \otimes \mathbb{C}[\Lambda_{ch}],$$

where $\mathfrak{h}_{ch,b}$ and $\mathfrak{h}_{ch,f}$ are copies of $\Lambda_{ch,\mathbb{C}}$ (see Section 5.1), does bear the structure of a lattice vertex algebra. Indeed, if $(\Lambda, q)$ is unimodular, then the condition $q(\lambda_z, \alpha) \in \mathbb{Z}, \forall \alpha \in \Lambda$ is equivalent to $\lambda_z \in \Lambda$. Hence denoting

$$\Lambda_{ch,\mathbb{R}} := \Lambda_{ch} \otimes_{\mathbb{Z}} \mathbb{R}, \qquad \Lambda_{ch,z} := \Lambda_{ch,\mathbb{R}} \cap \Lambda_z, \qquad \Lambda_{ch,\bar{z}} := \Lambda_{ch,\mathbb{R}} \cap \Lambda_{\bar{z}},$$

we get a decomposition

$$\Lambda_{ch,\mathbb{R}} = \Lambda_{ch,z} \oplus \Lambda_{ch,\bar{z}} \tag{21}$$

which is defined over $\mathbb{Z}$, i.e. $\Lambda_{ch} \cap \Lambda_z \subset \Lambda$ and $\Lambda_{ch,z} = (\Lambda_{ch} \cap \Lambda_z) \otimes_{\mathbb{Z}} \mathbb{R}$. Similarly for $\Lambda_{ch,\bar{z}}$. This gives rise to a lattice vertex algebra $V(\Lambda_{ch}, q, \Lambda_{ch,z})$ called the *chiral subalgebra* of $V(\Lambda, q, \Lambda_z)$. Its space of states is exactly $V_{ch}$. For example, in view of Lemma 3.8, the integral lattice $(\Lambda, q)$ and the $(z, \bar{z})$-decomposition associated to a torus as defined in (18) satisfy the conditions (a) and (b), and hence possess a chiral subalgebra. Further, the decomposition (21) allows us to exhibit the following simple structure of the chiral subalgebra:

**Proposition 5.4.** *Under the conditions* (a) *and* (b) *above, the chiral subalgebra $V(\Lambda_{ch}, q, \Lambda_{ch,z})$ is the tensor product of two commuting lattice vertex algebras in the sense of* [11]. *In particular, the fields in $V(\Lambda_{ch}, q, \Lambda_{ch,z})$ only contain integral powers in $z$ and $\bar{z}$.*

**Proof.** We already showed that the unimodularity of $(\Lambda, q)$ implies that $\Lambda_{ch} \cap \Lambda_z$ and $\Lambda_{ch} \cap \Lambda_{\bar{z}}$ are sublattices of $\Lambda$. Since $\Lambda_z$ is orthogonal to $\Lambda_{\bar{z}}$ with respect to $q$, the condition (b) implies that $q$ is non-degenerate on $\Lambda_z$ and $\Lambda_{\bar{z}}$. Moreover, since $\Lambda_{ch,z} = (\Lambda_{ch} \cap \Lambda_z) \otimes_{\mathbb{Z}} \mathbb{R}$ is contained in $\Lambda_z$, the $(z,\bar{z})$-decomposition of $\Lambda_{\mathbb{R}}$ induces the trivial decomposition on $\Lambda_{ch,z}$, i.e. $\Lambda_{ch,z}$ has no $\bar{z}$-part. Similarly, $\Lambda_{ch,\bar{z}}$ has no $z$-part. This gives rise to two lattice vertex algebras $V(\Lambda_{ch} \cap \Lambda_z, q, \Lambda_{ch,z})$ and $V(\Lambda_{ch} \cap \Lambda_{\bar{z}}, q, \Lambda_{ch,\bar{z}})$ in the sense of [11]. Since all the supercommutators between the $z$- and $\bar{z}$-parts are trivial (see (15)), we say that they commute with each other. Further, if we denote their space of states by $V_{ch,z}$ respectively $V_{ch,\bar{z}}$, then we have the tensor structure

$$V_{ch} = V_{ch,z} \otimes V_{ch,\bar{z}}.$$

Now it is clear that all the fields in the chiral subalgebra only contain integral powers in $z$ and $\bar{z}$. This completes the proof. $\square$

Note that if $\Lambda_{ch}$ is of maximal rank in $\Lambda$, then we have $V_{ch} = W$, which means that the chiral subalgebra contains exactly all the fields with solely integral powers in $z$ and $\bar{z}$. The maximality of its rank also yields the rationality of $V(\Lambda, q, \Lambda_z)$ (see Proposition 5.6). In order to give the definition of rationality, we explain now the module structure of a lattice vertex algebra $V(\Lambda, q, \Lambda_z)$ over its chiral subalgebra $V(\Lambda_{ch}, q, \Lambda_{ch,z})$.

The state-field correspondence on $V(\Lambda, q, \Lambda_z)$ induces the structure of an $\mathbb{R}^2$-fold algebra on itself, i.e. for every $(m, \bar{m}) \in \mathbb{R}^2$, we have an even morphism

$$V \otimes V \longrightarrow V, \qquad a \otimes b \longmapsto a_{(m,\bar{m})} b,$$

where $a_{(m,\bar{m})}$ is a coefficient of $Y(a, z, \bar{z}) = \sum_{(m,\bar{m}) \in \mathbb{R}^2} a_{(m,\bar{m})} z^{-m-1} \bar{z}^{-\bar{m}-1}$. This structure restricts to a $\mathbb{Z}^2$-fold module structure on $V(\Lambda, q, \Lambda_z)$ over its chiral subalgebra (see [20] for more details).

**Definition 5.5.** A lattice vertex algebra $V(\Lambda, q, \Lambda_z)$ is rational if it decomposes into a finite sum of irreducible modules over its chiral subalgebra. A $N = 2$ lattice vertex algebra is rational, if its underlying lattice vertex algebra (without the $N = 2$ structure) is rational.

It turns out that the rationality is determined by the size of the chiral sublattice. Indeed, there is an obvious decomposition of $V$. For any $\alpha \in \Lambda$, denote by $\chi_\alpha \subset \mathbb{C}[\Lambda]$ the subspace spanned by all vectors $\mathrm{e}^{\alpha+\lambda}$ with $\lambda \in \Lambda_{ch}$. Clearly, it is independent of the choice of the representant of $[\alpha] \in \Lambda/\Lambda_{ch}$, i.e. $\chi_\alpha = \chi_{\alpha'}$ for $[\alpha] = [\alpha'] \in \Lambda/\Lambda_{ch}$. Then

$$V = \bigoplus_{[\alpha] \in \Lambda/\Lambda_{ch}} \operatorname{Sym} \mathfrak{h}_b^{<} \otimes \bigwedge \mathfrak{h}_f^{<} \otimes \chi_\alpha =: \bigoplus_{[\alpha] \in \Lambda/\Lambda_{ch}} V_\alpha, \tag{22}$$

and each $V_\alpha$ is a $\mathbb{Z}^2$-fold module over the chiral subalgebra. We show

**Proposition 5.6.** *A lattice vertex algebra $V(\Lambda, q, \Lambda_z)$ is rational if and only if $\Lambda_{ch}$ is of maximal rank in $\Lambda$.*

**Proof.** If $\Lambda_{ch}$ is of maximal rank, i.e. $[\Lambda : \Lambda_{ch}] < \infty$, then the decomposition (22) is finite. We show that in this case, each $V_\alpha$ is an irreducible module. Indeed, as $\Lambda_{ch}$ is of maximal rank, $V_{ch}$ is isomorphic to $V_\alpha$ as vector space, and the action of $V_{ch}$ on $V_\alpha$ is faithful. In view of Proposition 5.4 and [11, Prop.5.4] we only need to check that if for some $v \in V_\alpha$, we have $(E^i_{-1} \otimes 1)_m v = 0$ and $(1 \otimes e^a)_m v = 0$, $\forall m$, $\forall i$ and $\forall a \in \Lambda_{ch}$ and similarly for the $\bar{z}$-part, then $v = 0$. This is clear by inspecting the explicit expressions of the corresponding field of these vectors (see the examples of fields given in Section 5.1).

Conversely, if $V(\Lambda, q, \Lambda_z)$ is rational, then the sum (22) must be finite, hence $\Lambda_{ch}$ is of maximal rank. □

For the lattice vertex algebra associated to tori, in view of Lemma 3.8, Proposition 5.6 has as consequence the following

**Theorem 5.7.** (i) *The lattice vertex algebra $V(\mathbb{T}, G, B)$ associated to a real torus with a constant metric $G$ and a $B$-field is rational if and only if $G$ and $B$ are both rational.*

(ii) *The $N = 2$ lattice vertex algebra $V(\mathbb{T}, \mathcal{I}, \mathcal{J})$ associated to a generalized complex torus is rational if and only if the composition $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$.*

(iii) *The $N = 2$ lattice vertex algebra $V(T, G, B)$ associated to a complex torus $T$ endowed with a constant Kähler metric $G$ and a $B$-field is rational if and only if $G$ and $B$ are both rational.*

**Remark 5.8.** In terms of the partition function $Z$, we see that the sum in (19) can be written as a finite sum over the elements of $\Lambda/\Lambda_{ch}$ if $V(\mathbb{T}, G, B)$ is rational. Indeed, due to the following decomposition over $\mathbb{Z}$:

$$\Lambda_{ch} = (\Lambda_{ch,z} \cap \Lambda) \oplus (\Lambda_{ch,\bar{z}} \cap \Lambda),$$

the sum becomes

$$\sum_{b \in \Lambda/\Lambda_{ch}} \left( \sum_{c \in \Lambda_{ch,z} \cap \Lambda} \mathsf{q}^{\frac{1}{2} q(c+b_z, c+b_z)} \right) \left( \sum_{d \in \Lambda_{ch,\bar{z}} \cap \Lambda} \bar{\mathsf{q}}^{\frac{1}{2} q(d+b_{\bar{z}}, d+b_{\bar{z}})} \right).$$

The first summation is finite if $\Lambda_{ch}$ is of maximal rank in $\Lambda$.

In the next section we draw some consequences of Theorem 5.7.

*5.4. Complex multiplication, rationality and mirror symmetry*

In this paragraph we use Theorem 5.7 to rephrase results of the first part of the paper in terms of lattice vertex algebras. Let us start out with an analogue of [13, Thm 5.4], which shows that mirror symmetry for generalized complex tori can be alternatively expressed in terms of their associated lattice vertex algebras. First we recall the definition of mirror symmetry for vertex algebras from [13].

**Definition 5.9.** Two $N = 2$ lattice vertex algebras are mirror partners if there is an isomorphism $f : V \to V'$ of their space of states, such that

(i) $f(|vac\rangle) = f(|vac'\rangle)$,
(ii) $fT = T'f$, $f\bar{T} = \bar{T}'f$,
(iii) for all $u, v \in V$, we have $Y'(f(u), z, \bar{z})v = f(Y(u, z, \bar{z})v)$,

with the additional property:

$$f(Q^+) = Q^{+'}, \qquad f(Q^-) = Q^{-'}, \qquad f(J) = J',$$
$$f(\bar{Q}^+) = \bar{Q}^{-'}, \qquad f(\bar{Q}^-) = \bar{Q}^{+'}, \qquad f(\bar{J}) = -\bar{J}'.$$

**Theorem 5.10.** *Two generalized complex tori $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ and $(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ are mirror partners if and only if the $N = 2$ lattice vertex algebras $V(\mathbb{T}, \mathcal{I}, \mathcal{J})$ and $V(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ are mirror partners.*

**Proof.** Let $\varphi$ be a mirror map between the two generalized complex tori. Since $\varphi$ preserves $q$ and the decomposition (9) (see Lemma 3.9(i)), it induces an isomorphism $f$ between the representations $\pi$ of $\hat{\mathfrak{h}}$ on $V$ and $\pi'$ of $\hat{\mathfrak{h}}'$ on $V'$, hence $f$ satisfies (i)–(iii) of Definition 5.9. As $\varphi$ sends $\mathcal{I} \mapsto \mathcal{J}'$, we have in view of (20)

$$Q^{\pm}_{\mathcal{I}} \mapsto Q^{\pm'}_{\mathcal{J}'} = Q^{\pm'}_{\mathcal{I}'} \qquad J_{\mathcal{I}} \mapsto J'_{\mathcal{J}'} = J'_{\mathcal{I}'}$$
$$\bar{Q}^{\pm}_{\mathcal{I}} \mapsto \bar{Q}^{\pm'}_{\mathcal{J}'} = \bar{Q}^{\mp'}_{\mathcal{I}'} \qquad \bar{J}_{\mathcal{I}} \mapsto \bar{J}'_{\mathcal{J}'} = -\bar{J}'_{\mathcal{I}'}$$

Similarly for $\mathcal{J} \mapsto \mathcal{I}'$, hence the $N = 2$ vertex algebra mirror morphism for both $N = 2$ structures.

Conversely, the isomorphism between the spaces of states induces a bijective map $\varphi : \Lambda \to \Lambda'$ of the lattices. The requirements (i)–(iii) of Definition 5.9 force $\varphi$ to be compatible with $q$ and $q'$. Finally, $\varphi$ maps $\mathcal{J} \mapsto \mathcal{I}'$ and $\mathcal{I} \mapsto \mathcal{J}'$ because of the $N = 2$ structure of the vertex algebras. This completes the proof. $\square$

Now we draw a direct consequence of Theorem 5.7, which shows that mirror symmetry has the virtue of letting the rationality of the lattice vertex algebra to be transmitted:

**Corollary 5.11.** *Suppose that $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ and $(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ are mirror generalized complex tori. Then the $N = 2$ lattice vertex algebra $V(\mathbb{T}, \mathcal{I}, \mathcal{J})$ is rational if and only if $V(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ is rational.*

**Proof.** From Theorem 5.7(ii) it follows that $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$. Lemma 3.8(i) implies that the decomposition (9) is defined over $\mathbb{Q}$. Hence the same holds for $(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ due to Lemma 3.9(i). $\square$

Further, by combining Theorems 2.5 and 5.7 one gets

**Corollary 5.12.** *An Abelian variety $X$ is of CM-type if and only if $X$ admits a rational $N = 2$ lattice vertex algebra $V(X, G, B)$.*

**Proof.** If $X$ is of CM-type, one can choose $B = 0$ together with the rational Kähler metric claimed by Theorem 2.5 to define a rational $N = 2$ lattice vertex algebra $V(X, G, B)$ in view of Theorem 5.7(iii). Conversely, the rationality of $V(X, G, B)$ forces $G$ to be rational again by Theorem 5.7(iii), and its $N = 2$ structure means that $G$ is Kähler. Again by Theorem 2.5, $X$ is of CM-type. $\square$

Finally, we give an answer to our question (Q) on the interplay between complex multiplication, rationality of the lattice vertex algebra and mirror symmetry for Abelian varieties. It is actually a reformulation of Theorem 3.11 and Proposition 4.1.

**Corollary 5.13.** *Let $(X, G, B)$ and $(X', G', B')$ be mirror Abelian varieties. If the $N = 2$ lattice vertex algebra $V(X, G, B)$ is rational, then $X$ and $X'$ are isogenous and both of CM-type. Conversely, however, there exist mirror Abelian varieties $(X, G, B)$ and $(X', G', B')$ such that $X$ and $X'$ are isogenous and both of CM-type, but neither $V(X, G, B)$ nor $V(X', G', B')$ is rational.*

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### Appendix. An isomorphism to $N = 2$ SCVA in [13]

We show Proposition 5.3. Recall from [13, Section 3] the

**Definition A.1.** Two $N = 2$ superconformal vertex algebras (SCVA) are isomorphic if there is an isomorphism $f : V \to V'$ of their space of states, such that

(i) $f(|vac\rangle) = f(|vac'\rangle)$
(ii) $fT = T'f$, $f\bar{T} = \bar{T}'f$

(iii) For all $u, v \in V$, we have $Y'(f(u), z, \bar{z}) f(v) = f(Y(u, z, \bar{z})v)$
(iv) $fL = L'$, $fQ^{\pm} = Q'^{\pm}$, $fJ = J'$, similarly for the $\bar{z}$-part.

Let $(T, G, B)$ be a complex torus. Recall the charge lattice isomorphism from [9]:

$$\phi := \frac{1}{\sqrt{2}} \begin{pmatrix} -G-B & 1 \\ G-B & 1 \end{pmatrix} : \Gamma_{\mathbb{R}} \oplus \Gamma_{\mathbb{R}}^* \longrightarrow \Gamma_{\mathbb{R}}^* \oplus \Gamma_{\mathbb{R}}^*$$

with inverse

$$\phi^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -G^{-1} & G^{-1} \\ 1 - BG^{-1} & 1 + BG^{-1} \end{pmatrix}.$$

Elementary calculations show that $\phi^{-1}$ is an isometry, i.e.:

$$q(\phi^{-1}\cdot, \phi^{-1}\cdot) = \begin{pmatrix} G^{-1} & 0 \\ 0 & -G^{-1} \end{pmatrix}. \tag{23}$$

Writing $\phi(a) =: (\phi_1(a), \phi_2(a))$, the decomposition $\Lambda_{\mathbb{R}} = \Lambda_z \oplus \Lambda_{\bar{z}} = \mathrm{Graph}_{\Gamma_{\mathbb{R}}}(-G+B) \oplus \mathrm{Graph}_{\Gamma_{\mathbb{R}}}(G+B)$ from (9) corresponds to

$$a \mapsto a_z := \phi^{-1} \circ \phi_1(a) \quad \text{and} \quad a \mapsto a_{\bar{z}} := -\phi^{-1} \circ \phi_2(a).$$

Write $f := \phi^{-1}$. We show that $f$ is the isomorphism we are looking for. Here we only give the calculation for the bosonic part. The fermionic part is similar.

We interpret [13]'s choice of bases as follows: $\{\alpha^i\}_{i=1\ldots 2g}$ as the basis of the *first* component of $\Gamma_{\mathbb{R}}^* \oplus \Gamma_{\mathbb{R}}^*$ and $\{\bar{\alpha}^i\}_{i=1\ldots 2g}$ as the basis of the *second* component of $\Gamma_{\mathbb{R}}^* \oplus \Gamma_{\mathbb{R}}^*$. Set

$$E^i := f(\alpha^i) \quad \text{and} \quad \bar{E}^i := f(\bar{\alpha}^i)$$

Then for the dual bases, i.e. $\{\widetilde{\alpha}^j\} \in \Gamma_{\mathbb{R}}^* \oplus 0$ and $\{\widetilde{\bar{\alpha}}^j\} \in 0 \oplus \Gamma_{\mathbb{R}}^*$ with $G^{-1}(\alpha^i, \widetilde{\alpha}^j) = G^{-1}(\bar{\alpha}^i, \widetilde{\bar{\alpha}}^j) = \delta^{ij}$ set

$$\widetilde{E}^i := f(\widetilde{\alpha}^i) \quad \text{and} \quad \widetilde{\bar{E}}^i := f(\widetilde{\bar{\alpha}}^i).$$

Then in view of (23) we have $q(E^i, \widetilde{E}^j) = \delta^{ij}$ and $q(\bar{E}^i, \widetilde{\bar{E}}^j) = -\delta^{ij}$. On the representations, $f$ induces the correspondence:

$$\alpha_s^i \mapsto E_s^i, \widetilde{\alpha}_s^i \mapsto \widetilde{E}_s^i \text{ for } s \in \mathbb{Z}^* \quad \text{and} \quad (G^{-1})^{kj} P_k \mapsto E_0^j.$$

Then the commutators are

$$\begin{aligned} [E_s^i, E_p^j] &\overset{(15)}{=} s\delta_{s,-p} q(E^i, E^j) \\ &\overset{(23)}{=} s\delta_{s,-p} (G^{-1})^{ij} \\ &= [\alpha_s^i, \alpha_p^j], \end{aligned}$$

where $s, p \in \mathbb{Z}^*$. Similarly for the $\bar{z}$-part.

At this stage it is clear that $f$ possesses the properties (i), (ii) and (iv) of Definition A.1. For (iii) we first translate the notations in [13] into ours. For $(w, m) \in \Gamma \oplus \Gamma^*$:

$$\begin{aligned} &P_i(w, m) = \phi_{1i}(w, m) \quad \text{is the } i\text{-th coordinate of } \phi_1(w, m) \in \Gamma^*, \\ &\bar{P}_i(w, m) = \phi_{2i}(w, m) \quad \text{is the } i\text{-th coordinate of } \phi_2(w, m) \in \Gamma^*, \\ &k = \phi_1(w, m), \\ &\bar{k} = \phi_2(w, m). \end{aligned}$$

Then for $a = (w, m), b = (w', m') \in \Gamma \oplus \Gamma^*$, we have again by (23) $q(a_z, b_z) = G^{-1}(k, k')$ and $q(a_{\bar z}, b_{\bar z}) = -G^{-1}(\bar k, \bar k')$, and

$$\partial^s X(z) = \partial^{s-1}(G^{-1})^{jk} P_k \frac{1}{z} - \partial^s Y^j(z) \longmapsto \partial^{s-1} \sum_{m\in\mathbb{Z}} E_m z^{-m-1},$$

$$k_j Y^j(z)_+ = k_j \sum_{m<0} \frac{\alpha_m^j}{m z^m} \longmapsto \phi_{1j}(a) \sum_{m<0} \frac{E_m^j}{m z^m} = \sum_{m<0} \frac{\phi^{-1}(\sum_j \phi_{1j}(a)\alpha^j)_m}{m z^m} = \sum_{m<0} \frac{a_{zm}}{m z^m}.$$

Compare the state-field correspondence (17) with the one given in [13], the isomorphism is then obvious.

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